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The Structure of Certain Artinian Rings with Zero Singular Ideal

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INTRODUCTION

Let R be a ring with identity and M and N (left) R-modules with $M \subset N$. Then M is said to be *essential* in N if M intersects every nonzero submodule of N nontrivially. A left ideal I of R is called an essential left ideal if I is essential in R. We denote the *singular submodule* of N by $Z(N)$; $Z(N) = \{n \in N \mid In = (0)$ for some essential left ideal I of $R\}$. We shall assume throughout this paper that $Z(R) = 0$. In Section 1 we outline the construction of Johnson's ring of quotients [3] and some of its properties and develop certain other preliminary results. In Section 2 we determine the structure of Artinian rings with zero singular ideal having the property that each principal indecomposable left ideal contains a unique minimal left ideal. In particular we show that if R is such a ring then R is a certain distinguished subring of a complete blocked triangular matrix ring over a division subring of R. We also consider the case where R is a left generalized uniserial ring and generalize results of A. W. Goldie [2] and I. Murase [5]. Finally we apply the methods to obtain the structure of finite-dimensional algebras with zero singular ideal.

1. PRELIMINARIES

If R is a ring with zero singular ideal, then Johnson [3] has constructed a ring extension Q of R which we shall call Johnson's ring of quotients of R. Essentially, the definition of Q in [3] was as follows: $Q = \bigcup_A \mathrm{Hom}_R(A, R)$, where A is an essential left ideal of R. If $x, y \in Q$, then we take $x = y$ if and only if $xa = ya$ for every a in some essential left ideal

$$A \subset (\text{domain } x) \cap (\text{domain } y).$$

This extension Q of R is a ring of quotients of R in the sense that ${}_RQ$ is an essential extension of ${}_RR$ and since $Z({}_RR) = 0$ so does $Z({}_RQ)$. The following property of Q is established by Lambek [4].

LEMMA 1.1. *If I and J are left ideals of R with I essential in R, then* $\mathrm{Hom}_R(I, J)$ is isomorphic to

$$J : I = \{q \in Q \mid Iq \subset J\}$$

under the correspondence $(i \to iq) \leftrightarrow q$, $q \in Q$, $i \in I$.

LEMMA 1.2. *Suppose that M and N are submodules of ${}_RQ$ and $f \in \mathrm{Hom}_R(M, N)$. Then there exists a unique $\bar{f} \in \mathrm{Hom}_Q(QM, QN)$ such that $\bar{f}|_M = f$. If f is one-to-one or onto, so is $\bar{f}$.*

Proof. For $x = \sum q_i m_i \in QM$, define $\bar{f}(x) = \sum q_i f(m_i)$. The assertions of the lemma are easily verified.

We denote the left and right annihilators of a subset $S \subset R$ by S_l and S_r, respectively.

LEMMA 1.3. *If I is a left ideal of R, then $I_r = \{x \in R \mid x_l \cap I$ is essential in $I\}$.*

Proof. $x \in I_r \Rightarrow x_l \supset I \Rightarrow x_l \cap I$ is essential in I. Conversely, if $x_l \cap I$ is essential in I, then $(x_l \cap I : y)\, yx = 0$ for all $y \in I$. But $(x_l \cap I : y)$ is essential in R. For, if $(x_l \cap I : y) \cap I' = 0$, then $I'y \cap (x_l \cap I) = 0$ so $I'y = 0$. Thus $I' \subset (x_l \cap I : y) \cap I' = 0$ and $I' = 0$. Now since $(x_l \cap I : y)\, yx = 0$, we get $yx = 0$. Hence $x \in I_r$.

We say that M is a *uniform* R-module if every nonzero submodule of M is an essential submodule of M.

LEMMA 1.4. *If M is a uniform R-module, I is a left ideal of R and $0 \neq \alpha \in \mathrm{Hom}_R(M, I)$, then α is one-to-one.*

Proof. If $\ker \alpha \neq 0$ then for any $m \in M$, $(\ker \alpha : m)$ is an essential left ideal of R. Furthermore,

$$(\ker \alpha : m)\, \alpha(m) = \alpha\, (\ker \alpha m) = 0$$

and hence $\alpha(m) = 0$ since $Z(R) = 0$. Thus $\alpha = 0$, a contradiction.

LEMMA 1.5 (Dual Bases Lemma, [1], Chapter VII, Prop. 3.1). *An R-module M is projective if and only if there exists a family $\{a_\alpha\} \subset M$ and a family $\{\lambda_\alpha\} \subset \mathrm{Hom}_R(M, R)$ such that, for each $m \in M$, $m = \sum_\alpha \lambda_\alpha(m)\, a_\alpha$ where $\lambda_\alpha(m) = 0$ for all but a finite number of indices α.*

For the rest of this section we shall assume that R is left Artinian. We denote the (left) socle of R by E. Then E is the unique minimal essential left ideal of R.

LEMMA 1.6. *The socle E of R is a projective left R-module.*

Proof. Let M be a minimal left ideal of R. Since $EM \neq 0$, there exists a minimal left ideal $I \subset E$ such that $IM \neq 0$ and hence $IM = M$. Since I is not nilpotent, $I = Re$ with $e^2 = e$ and $M \cong I$ is projective. Thus E is clearly projective.

PROPOSITION 1.7. $Q = QE$.

Proof. Since E is projective, Lemma 1.5 implies that there exists $\{a_\alpha\} \subset E$ and $\{\lambda_\alpha\} \subset \mathrm{Hom}_R(E, R)$ such that

$$y = \textstyle\sum_\alpha \lambda_\alpha(y)\, a_\alpha$$

for all $y \in E$. By Lemma 1.1 there exist $q_\alpha \in Q$ such that $\lambda_\alpha(y) = yq_\alpha$ for all $y \in E$. Hence $y = \sum_\alpha yq_\alpha a_\alpha$ and so $E(1 - \sum_\alpha q_\alpha a_\alpha) = 0$. Thus, since $Z(R) = 0$, $1 = \sum_\alpha q_\alpha a_\alpha \in QE$.

2. THE STRUCTURE THEOREM

We shall assume throughout the rest of this paper that R is a left Artinian ring with zero singular ideal having the property that every principal indecomposable left ideal contains a unique minimal left ideal.

LEMMA 2.1. *If R has the above properties, then R can be written as a direct sum of indecomposable (two-sided) ideals with the above properties as rings.*

Proof. Let e_i be primitive idempotents and let M_i be the unique minimal left ideal of Re_i for $i = 1, 2$. If $M_1 \ncong M_2$, then we must have $M_1Re_2 = 0$ and hence if follows from Lemma 1.3 that $Re_1Re_2 = 0$. Similarly, $Re_2Re_1 = 0$. Thus if we write R as a direct sum of indecomposable left ideals and group these according to the isomorphism type of their unique minimal ideals, we clearly obtain a decomposition of R as a direct sum of ideals which as rings have the above properties. Since each of these rings clearly has homogeneous socle they are indecomposable.

We shall assume henceforth that R is indecomposable.

LEMMA 2.2. *The irreducible left ideals of R are R-isomorphic and R has an irreducible summand.*

Proof. This is immediate from the reasoning of the above lemma and Lemma 1.6.

Let $e_1, \ldots, e_m$ be a complete set of primitive idempotents of R. We say that $e_i \sim e_j$ if and only if $Re_i \cong Re_j$, for all $1 \leqslant i, j \leqslant m$. This is clearly an equivalence relation. We partially order the resulting equivalence classes by saying that $[e_i] \leqslant [e_j]$ if and only if Re_i is isomorphic to a submodule of Re_j. This is a partial order since R is left-Artinian. Note that this partially ordered set contains a unique minimal element, namely, the equivalence class determined by the idempotent corresponding to an irreducible summand of R.

LEMMA 2.3. $[e_i] \leqslant [e_j]$ *if and only if* $e_i R e_j \neq 0$.

Proof. The necessity is obvious and the sufficiency follows easily from Lemma 1.4.

LEMMA 2.4. *Let $e_1, \ldots, e_m$ be a complete set of primitive orthogonal idempotents of R. Then there exists an integer k, k integers $n_1, n_2, \ldots, n_k$ with $\sum_{i=1}^{k} n_i = m$ and an indexing $e_j{}^i$ of the above idempotents with $i = 1, \ldots, k$ and $1 \leqslant j \leqslant n_i$ such that $e_j{}^i \sim e_{j'}^i$, for all i, j, and j' and $e_{j'}^{i'} R e_j{}^i = 0$ if $i' > i$, for all j and j'.*

Proof. Let k be the number of equivalence classes of idempotents resulting from the above equivalence relation. First number the equivalence classes from 1 to k in such a way that if $[e_j]_{i'} \leqslant [e_{j'}]_i$ then $i' \geqslant i$. Now let n_i be the number of idempotents in the ith class. If we index the idempotents of the ith class with a superscript i and a subscript ranging from 1 to n_i, then it is immediate from the definition of $\sim$ and Lemma 2.3 that this is the desired indexing.

LEMMA 2.5. *$Re_j{}^i$ is irreducible if and only if $i = k$.*

Proof. This is immediate from Lemma 2.2 and the manner in which the idempotents were indexed above.

LEMMA 2.6. *For all i, j, $e_j{}^i R e_j{}^i$ is a division ring.*

Proof. Let N denote the radical of R. It suffices to show that $e_j{}^i N e_j{}^i = 0$. If this were not so then Lemma 1.4 would imply that $Re_j{}^i$ is isomorphic to a submodule of $Ne_j{}^i$.

LEMMA 2.7. *$Q = Qe_1^1 \oplus \cdots \oplus Qe_{n_1}^1 \oplus \cdots \oplus Qe_1^k \oplus \cdots \oplus Qe_{n_k}^k$, each Qe_j^i is an irreducible left ideal in Q and $Qe_j^i \cong Qe_{j'}^{i'}$ for all i, i', j, and j'. In particular Q is a simple ring.*

Proof. Let M_j^i denote the unique irreducible submodule of Re_j^i. Then by Proposition 1.7,

$$\begin{aligned} Q &= QE \\ &= QM_1^1 \oplus \cdots \oplus QM_{n_1}^1 \oplus \cdots \oplus QM_1^k \oplus \cdots \oplus QM_{n_k}^k \\ &\subset Qe_1^1 \oplus \cdots \oplus Qe_{n_1}^1 \oplus \cdots \oplus Qe_1^k \oplus \cdots \oplus Qe_{n_k}^k = Q. \end{aligned}$$

Hence $QM_j^i = Qe_j^i$ for each i, j. Since $M_j^i \cong_R M_{j'}^{i'}$, Lemma 1.2 implies that $Qe_j^i \cong_Q Qe_{j'}^{i'}$. Also, since ${}_RM_j^i$ is essential in ${}_R(Qe_j^i)$ and ${}_RM_j^i$ is irreducible, M_j^i is contained in every nonzero Q-submodule of Qe_j^i. Thus every nonzero Q-submodule of Qe_j^i contains $QM_j^i = Qe_j^i$. Thus Qe_j^i is an irreducible Q-module.

LEMMA 2.8.

$$\mathrm{Hom}_Q (Qe_1^k, Qe_1^k) \cong e_1^kQe_1^k = e_1^kRe_1^k \cong \mathrm{Hom}_R (Re_1^k, Re_1^k).$$

Proof. Since Re_1^k is irreducible, it is the socle of Qe_1^k. Thus $Re_1^kqe_1^k \subset Re_1^k$, for all $q \in Q$. Thus there exists $r \in R$ such that $e_1^kre_1^k = e_1^kqe_1^k$.

THEOREM 2.9. *There exists a positive integer k, and positive integers $n_1, n_2, \ldots, n_k$ such that R is isomorphic to a blocked triangular matrix ring such that the i, jth block is an arbitrary $n_i \times n_j$ matrix with entries in an additive subgroup D_{ij} of a fixed division subring D of R. These subgroups satisfy the following additional properties:*

(1) *$D_{il}D_{lj} \subset D_{ij}$, for all i, j, l.*

(2) *D_{ii} is a division ring for each i.*

(3) *$D_{ki} = D$ for all i.*

(4) *D_{ij} is finite dimensional as a left vector space over D_{ii} for all i and j.*

Conversely, if D is a fixed division ring, k and $n_1, \ldots, n_k$ fixed integers and D_{ij} are fixed subgroups of D with the above properties then the corresponding blocked triangular matrix ring is a left Artinian ring with zero singular ideal and such that each principal indecomposable left ideal contains a unique minimal left ideal.

Proof. Let $e_1, \ldots, e_m$ be a complete set of primitive orthogonal idempotents for R, k a positive integer, $n_1, \ldots, n_k$ positive integers, and e_j^i an indexing of these idempotents as described in Lemma 2.4. Let g_t^i be a fixed isomorphism of Re_t^i onto Re_1^i and let

$$r_t^i = g_t^i(e_t^i) \in e_t^i R e_1^i,$$

and

$$s_t^i = (g_t^i)^{-1}(e_1^i) \in e_1^i R e_t^i.$$

A computation shows that

$$r_t^i s_t^i = e_t^i \qquad \text{and} \qquad s_t^i r_t^i = e_1^i. \tag{$*$}$$

Let h^i be a fixed isomorphism of Qe_1^i onto Qe_1^k with h^k equal the identity on Qe_1^k,

$$\alpha^i = h^i(e_1^i) \in e_1^i Q e_1^k,$$

and

$$\beta^i = (h^i)^{-1}(e_1^k) \in e_1^k Q e_1^i.$$

A computation shows that

$$\alpha^i \beta^i = e_1^i \qquad \text{and} \qquad \beta^i \alpha^i = e_1^k. \tag{$**$}$$

We define a function θ from R into the ring of $m \times m$ blocked triangular matrices with entries in $D = e_1^k R e_1^k$ by letting $\theta(a)$ be the matrix whose (i, j)th block is the $n_i \times n_j$ matrix

$$A_{ij} = (\beta^i s_u^i e_u^i a e_v^j r_v^j \alpha^j) = (\beta^i s_u^i a r_v^j \alpha^j).$$

The entries in this block belong to

$$D_{ij} = \{\beta^i x \alpha^j \mid x \in R\},$$

which is an additive subgroup of $e_1^k Q e_1^k = e_1^k R e_1^k$, isomorphic to $e_1^i R e_1^j$. Since this is a ring isomorphism when $i = j$, D_{ii} is isomorphic to $e_1^i R e_1^i$ which is a division ring.

We will now show that θ maps onto the blocked triangular matrix ring whose (i, j)th block has arbitrary entries in D_{ij} by showing that there exists an element $\bar{x} \in R$ such that $(A_{ij})_{uv} = \beta^i x \alpha^j$ for any i, j, u, v and any $x \in R$, and all other entries of $\theta(\bar{x})$ are zero. Let $\bar{x} = r_u^i x s_v^j \in e_u^i R e_v^j$. Then since the $e_{j'}^{i'}$ are orthogonal idempotents, all the entries of $\theta(\bar{x})$ are zero except the (u, v)th entry of the (i, j)th block, while

$$\begin{aligned}(A_{ij})_{uv} &= \beta^i s_u^i r_u^i x s_v^j r_v^j \alpha^j \\ &= \beta^i e_1^i x e_1^j \alpha^j = \beta^i x \alpha^j.\end{aligned}$$

Clearly from the choice of h^k, $\alpha^k = \beta^k = e_1{}^k$ and so $D_{kk} = e_1{}^k R e_1{}^k = D$. Also $D_{kj} \subset D_{kk}$. But for any j, $e_1{}^k x \beta^j \in e_1{}^k R e_1{}^k Q e_1{}^j \subset e_1{}^k R e_1{}^k Q \subset EQ$ since $Re_1{}^k$ is an irreducible R-module (Lemma 2.5). But $EQ \subset R$ since ${}_RQ$ is an essential extension of ${}_RR$ and E is the unique minimal essential left ideal of R. Thus $e_1{}^k x \beta^j \in R$ and so $e_1{}^k x \beta^j \alpha^j = e_1{}^k x e_1{}^k \in D_{kj}$. Thus $D_{kj} \supset D_{kk}$.

One computes that θ is a ring homomorphism and that $D_{il} D_{lj} \subset D_{ij}$ for all i and j using conditions (*) and (**). For any $0 \neq a \in R$, $e_j{}^i a e_{j'}^{i'} \neq 0$ for some i, i', j, j', and hence it follows from (*) and (**) that $\theta(a) \neq 0$. Thus θ is one-to-one.

We have, therefore, established an isomorphism between R and a ring of matrices of the type described in the theorem and have verified the necessity of (1)-(3). Condition (4) is most easily verified by noting that the matrix ring and hence also R would not be left-Artinian if (4) did not hold.

The converse of the theorem follows readily from an examination of matrix rings of the type described above.

We say that R is *left generalized uniserial* if each principal indecomposable left ideal of R has a unique composition series.

COROLLARY 2.10. *R is an (indecomposable) left generalized uniserial ring (with zero left singular ideal) if and only if R has a matrix representation of the type described in the preceding theorem which has the following additional properties:*

(1) *D_{ij} has dimension at most one as a (left) vector space over D_{ii} for all i and j.*

(2) *If $D_{ij} \neq 0$ and $D_{i'j} \neq 0$ with $i \leqslant i'$, then $D_{i'i} \neq 0$.*

Proof. A left generalized uniserial ring satisfies the hypothesis of Theorem 2.9 and hence has a matrix representation of the type described there. That (1) and (2) are necessary and sufficient for such a matrix ring to be left generalized uniserial follows from a straightforward examination of these matrix rings.

We say that R is *meet irreducible* if and only if any two nonzero ideals of R have nonzero intersection.

COROLLARY 2.11. *R is a meet irreducible left generalized uniserial ring (with zero left singular ideal) if and only if R has a matrix representation of the type described in Theorem 2.9 with the additional property that D_{ij} is one-dimensional as a left vector space over D_{ii} for all i and j with $j \leqslant i$.*

LEMMA 2.12. *Let A be a finite-dimensional algebra over a field K. Then A has zero left singular ideal if and only if A is isomorphic to a subalgebra of a finite-dimensional K-algebra $\hat{A}$ with the following properties:*

(1) *Each principal indecomposable left ideal of* $\hat{A}$ *contains a unique minimal left ideal.*

(2) $\hat{A}$ *has zero left singular ideal.*

(3) *The socles of* ${}_A A$ *and* ${}_{\hat{A}}\hat{A}$ *coincide.*

Proof. Let Q be Johnson's (left) ring of quotients for A. Johnson [3] has shown that Q is a semisimple ring with minimum condition. This can also be seen by modifying slightly the proof of Lemma 2.7. We note that Q is a finite-dimensional algebra over K since ${}_A Q$ is the injective envelope of ${}_A A$.

Let $e_1, \ldots, e_m$ be a complete set of primitive orthogonal idempotents for A. We can write each e_i as

$$e_i = e_i^1 + \cdots + e_i^{k_i},$$

so that the

$$\{e_i^j \mid j = 1, \ldots, k_i;\ i = 1, \ldots, m\}$$

is a complete set of primitive orthogonal idempotents for Q. Note that k_i is the number of summands in any irreducible decomposition of the socle of Ae_i. For, let $M_1^i \oplus \cdots \oplus M_{u_i}^i$ be the socle of Ae_i. Then, by Proposition 1.6,

$$\begin{aligned} Q = QE &= QM_1^1 \oplus \cdots \oplus QM_{u_1}^1 \oplus \cdots \oplus QM_1^m \oplus \cdots \oplus QM_{u_m}^m \\ &= Qe_1 \oplus \cdots \oplus Qe_m . \end{aligned}$$

Hence $Qe_i = QM_1^i \oplus \cdots \oplus QM_{u_i}^i$ and so $u_i = k_i$ since the QM_j^i are irreducible Q-modules. Thus the socle of A contains $t = \sum_{i=1}^m k_i$ components. Note also that Ae_i contains a unique minimal submodule if and only if $k_i = 1$.

Let $\hat{A}$ be the smallest subalgebra of Q which contains A and $\{e_i^j \mid j = 1, \ldots, k_i;\ i = 1, \ldots, m\}$. Since Q is a finite-dimensional K-algebra so is $\hat{A}$. Let I be an irreducible left ideal of A. Then ${}_A I$ is projective by Lemma 1.6 and so $e_i I \neq 0$ implies that Ae_i is irreducible. Suppose $e_i^j I \neq 0$. Then $0 \neq e_i^j I = e_i^j e_i I$ and so Ae_i is an irreducible projective, $k_i = 1$, and $e_i^j = e_i$. Thus $e_i^j I \subset I$. Now any element of $\hat{A}$ is a sum of products of terms of the form ae_i^j with $a \in A$ and so $\hat{A} I \subset I$. Thus irreducible submodules of ${}_A A$ are irreducible submodules of ${}_{\hat{A}}\hat{A}$. Furthermore, for all i, j, $\hat{A}e_i^j$ is an A-submodule of Qe_i^j and hence contains a unique minimal left ideal of A, and hence also a unique minimal left ideal of $\hat{A}$. Since there are t irreducible components in the socle of ${}_A A$, the socles of ${}_A A$ and ${}_{\hat{A}}\hat{A}$ must coincide. Thus since ${}_A \hat{A}$ is an essential extension of ${}_A A$, ${}_{\hat{A}}\hat{A}$ has zero left singular ideal. The converse is immediate.

THEOREM 2.13. *Let* A *be a finite-dimensional algebra over a field* K. *Then*

A has zero left singular ideal if and only if A is (isomorphic to) a subalgebra of an algebra T with the following properties:

(1) *$T = T_1 \oplus \cdots \oplus T_n$, where each T_i is a two sided ideal of T and T_i is a complete blocked triangular matrix ring over a finite dimensional division algebra D_i over K.*

(2) *The socles of ${}_AA$ and ${}_TT$ coincide.*

Proof. This theorem is immediate from Lemma 2.12 and Theorem 2.9.

Remark. We have used the hypothesis that A is a finite-dimensional algebra in the proof of Lemma 2.12 only to show that $\hat{A}$ is a ring with left minimum condition. Thus Theorem 2.13 holds (with appropriate modifications) for any left Artinian ring R with zero singular ideal such that $\hat{R}$ is left Artinian. In particular the theorem holds whenever the injective envelope of ${}_RR$ is a finitely generated left R-module.

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